THE SUMS OF POWERS OF INTEGERS

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The problem of finding the sums of the powers of the integers from 1 to n has interested mathematicians for a long time. Expressions involving Bernoulli's numbers have been developed for $S_p(n)$ where

$$S_p(n) = 1^p + 2^p + 3^p + \cdots + n^p \tag{1}$$

with p a positive integer, as well as for sums of powers of the odd integers, from 1 to $2n-1$. For a review of the previous work in this field the reader is referred to Bachmann's *Niedere Zahlentheorie*, Second Part, pp. 16 ff., and to Nielsen, *Traité des Nombres de Bernoulli*, Chap. XVI.* As far as the writer is aware, the formulas for $S_p(n)$ and similar sums have always been derived by methods involving Bernoulli's numbers. In the present paper formulas are derived for $S_p(n)$ and related expressions by methods involving nothing more than the binomial theorem. A natural independent variable in terms of which to express $S_p(n)$ is the triangular number $n(n+1)/2 \equiv m = S_1(n)$. When p is odd, $S_p(n) = f_p(m)$ and when p is even, $S_p(n) = (2n+1)g_p(m)$, where f_p and g_p are polynomials with rational coefficients of degrees $(p+1)/2$ and $p/2$ respectively.

It is shown, furthermore, that $S_p(n) = F_p(n+1/2)$ where F_p is a polynomial with rational coefficients of degree $p+1$. When p is odd only even powers of

* Cf. also Schwatt, *An Introduction to the Operations with Series*, Ch. 5, Philadelphia, 1924.

$(n+1/2)$ occur in F_p; when p is even only odd powers of $(n+1/2)$ occur, with the exception of $S_0(n)=(n+1/2)-1/2$.

Let

$$R_p(2n-1) = 1^p + 3^p + 5^p + \cdots + (2n-1)^p. \tag{2}$$

It will be shown that

$$R_p(2n-1) = G_p(n),$$

where G_p is a polynomial with rational coefficients of degree $p+1$. When p is odd, G_p contains only even powers of n; when p is even, G_p contains only odd powers of n.

It is easily shown that the following relation holds:

$$R_p(2n-1) = S_p(2n-1) - 2^p S_p(n-1). \tag{2a}$$

We now proceed to establish*

THEOREM I:

$$S_{2p-1}(n) = \sum_{k=2}^{p} A_{pk} m^k, \tag{3}$$

where A_{pk} are rational numbers† *independent of n that satisfy*

$$A_{pp} = \frac{2^{p-1}}{p} \tag{4}$$

and the recursion formula

$$A_{pk} = -\frac{1}{p}\sum_{j=\mu}^{p-1}\binom{p}{2p-2j+1}A_{jk}, \qquad k < p. \tag{5}$$

Here μ is the least value which j can assume in (5) without making the binomial coefficient on the right side of the equation zero. This theorem is valid for all positive integral values of p except 1 in which case $S_1(n)=m$.

Proof.

$$\left[\frac{r(r+1)}{2}\right]^p - \left[\frac{r(r-1)}{2}\right]^p = \frac{2}{2^p}\sum_{k \text{ odd}}^{p}\binom{p}{k} r^{2p-k}$$
$$= \sum_{j=\mu}^{p} \frac{1}{2^{p-1}}\binom{p}{2p-2j+1} r^{2j-1}.$$

Summing for r from 1 to n, we have

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* Theorems I, II, III and IV are obtained, with the aid of Bernoulli numbers, in Chap. XVI of Nielsen's *Nombres de Bernoulli*, and theorems V and VI in Chap. I of Bachmann, *Niedere Zahlentheorie*, Second Part, p. 26.

† Equation (3) of course implies that $A_{jk}=0$ for $k>j$. This is also true of the coefficients B_{jk}, C_{jk}, D_{jk}, E_{jk} and F_{jk} which occur in Theorems II, III, IV, V and VI respectively.

$$
(6)\qquad \begin{aligned} m^p &= \sum_{j=\mu}^{p} \frac{1}{2^{p-1}}\binom{p}{2p-2j+1} S_{2j-1}(n) \\ &= \frac{p}{2^{p-1}} S_{2p-1}(n) + \frac{1}{2^{p-1}} \sum_{j=\mu}^{p-1}\binom{p}{2p-2j+1} S_{2j-1}(n). \end{aligned}
$$

Henceforth, the proof rests on mathematical induction. Assume that (3) holds if p is replaced by j, for $j=2, 3, \cdots (p-1)$. Then we shall show that $S_{2p-1}(n)$ also has the form (3), and obtain recursion formulas for the coefficients.

Replacing p by j in (3), substituting the result in (6) and solving for $S_{2p-1}(n)$, we have

$$
(7)\qquad \begin{aligned} S_{2p-1}(n) &= \frac{2^{p-1}}{p} m^p - \frac{1}{p} \sum_{j=\mu}^{p-1}\binom{p}{2p-2j+1} \sum_{k=2}^{j} A_{jk} m^k \\ &= \frac{2^{p-1}}{p} m^p - \frac{1}{p} \sum_{k=2}^{p-1}\sum_{j=\mu}^{p-1}\binom{p}{2p-2j+1} A_{jk} m^k. \end{aligned}
$$

It is seen that $S_{2p-1}(n)$ has the form (3).

Since $S_1(n)$ contains the first power and only the first power of m, it is essential to the proof that $S_1(n)$ shall never occur in (6), for any value of p considered in the proof, i.e., for $p=3, 4 \cdots$. It is easily seen that this condition is fulfilled since even in the most unfavorable case, namely, when $p=3$, the value of μ is 2. Therefore, in equation (6), the sum of lowest order which occurs is $S_{2\mu-1}(n) = S_3(n)$.

Since, now, $S_3(n) = m^2$, a formula which is easily demonstrated, equation (7) permits us to conclude that $S_5(n)$ and hence in general, $S_{2p-1}(n)$, has the form given in equation (3). Theorem I therefore follows by mathematical induction. Formulas (4) and (5) are now obtained by comparison of equations (7) and (3).

THEOREM I(A): *The coefficients A_{pk} can be expressed in the following determinant form:*

$$
(8)\qquad A_{pk} = \frac{(-1)^{p-k}2^{k-1}(k-1)!}{p!}\Delta_{pk},
$$

where

$$
(9)\qquad \Delta_{pk} = \left| a_{ij}^{pk} \right|,
$$

i and j take on the values $1, 2, 3, \cdots (p-k)$ *and*

$$
(9a)\qquad a_{ij}^{pk} = \binom{p-j+1}{2i-2j+3}.
$$

All of the elements of (9) are zero for $j > i+1$.

This is valid for $k<p$. For $k=p$ the equation (8) gives the correct result if the determinant (9) is arbitrarily assigned the value 1.

Proof. Equations (4) and (5) can be written in the following form:

$$\sum_{j=\mu}^{p}\binom{p}{2p-2j+1}A_{jk} = 2^{p-1}\delta_{pk}. \tag{10}$$

Here

$$\delta_{pk} = 1 \text{ if } p = k$$

$$\delta_{pk} = 0 \text{ if } p \neq k.$$

The A_{pk} can be obtained by solving the following $p-k+1$ equations from the set (10)

$$\begin{aligned}
\binom{p}{1}A_{p,k} + \binom{p}{3}A_{p-1,k} + \binom{p}{5}A_{p-2,k} + \cdots + \binom{p}{2p-2k+1}A_{kk} &= 0\\
\binom{p-1}{1}A_{p-1,k} + \binom{p-1}{3}A_{p-2,k} + \cdots + \binom{p-1}{2p-2k-1}A_{kk} &= 0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
\binom{k+2}{1}A_{k+2,k} + \binom{k+2}{3}A_{k+1,k} + \binom{k+2}{5}A_{kk} &= 0\\
\binom{k+1}{1}A_{k+1,k} + \binom{k+1}{3}A_{kk} &= 0\\
\binom{k}{1}A_{kk} &= 2^{k-1}.
\end{aligned} \tag{11}$$

Solving these equations by Cramer's rule and interchanging rows and columns in the determinants, we have the result given in equations (8) and (9).

THEOREM I(B): *The coefficients A_{pk} satisfy the following recursion formula*

$$A_{pk} = -\frac{1}{k}\sum_{i=k+1}^{2k-1} 2^{k-i}A_{pi}\binom{i}{2i-2k+1}. \tag{12}$$

This enables one to compute successively $A_{p,p-1}$, $A_{p,p-2}$, etc., from A_{pp} given by equation (4).

Proof. In formula (3) substitute $n-1$ for n and subtract the result from (3). This gives:

$$n^{2p-1} = \sum_{k=2}^{p}\frac{A_{pk}}{2^k}\left\{n^k\left[(n+1)^k-(n-1)^k\right]\right\}. \tag{13}$$

Expanding and rearranging terms on the right, and equating coefficients of powers of n, we obtain equations (4) and (12).

It is interesting to note that one can also obtain the determinant form (8) and (9) by starting with (12) instead of (10).

THEOREM II:

$$S_{2p}(n) = (2n+1)\sum_{k=1}^{p} B_{pk}m^k, \tag{14}$$

where B_{pk} are rational numbers independent of n that satisfy the relation

$$B_{pp} = \frac{2^{p-1}}{2p+1} \tag{15}$$

and the recursion formula

$$B_{pk} = -\frac{1}{2p+1}\sum_{j=\lambda}^{p-1}\left[2\binom{p}{2p-2j+1}+\binom{p}{2p-2j}\right]B_{jk}, \qquad k<p. \tag{16}$$

Here λ is the least value of j for which the bracket in (16) does not vanish.

In this case, p can have any integral value greater than or equal to 1.

Proof.

$$(2r+1)\left[\frac{r(r+1)}{2}\right]^p - (2r-1)\left[\frac{r(r-1)}{2}\right]^p$$
$$= \frac{1}{2^p}\sum_{j=\lambda}^{p}\left[4\binom{p}{2p-2j+1}+2\binom{p}{2p-2j}\right]r^{2j}.$$

Summing r from 1 to n, we obtain

$$(2n+1)m^p = \frac{1}{2^p}\sum_{j=\lambda}^{p}\left[4\binom{p}{2p-2j+1}+2\binom{p}{2p-2j}\right]S_{2j}(n).$$

Solving for $S_{2p}(n)$,

$$\begin{aligned} S_{2p}(n) &= \frac{2^p}{4p+2}(2n+1)m^p \\ &\quad - \frac{1}{4p+2}\sum_{j=\lambda}^{p-1}\left[4\binom{p}{2p-2j+1}+2\binom{p}{2p-2j}\right]S_{2j}(n). \end{aligned} \tag{17}$$

Assuming (14) to hold if p is replaced by j, for $j=1, 2, 3, \cdots, p-1$, substituting in (17) and reversing the order of summation, we obtain

$$\begin{aligned} S_{2p}(n) &= \frac{2^{p-1}}{2p+1}(2n+1)m^p \\ &\quad - \frac{1}{2p+1}\sum_{k=1}^{p-1}\sum_{j=\lambda}^{p-1}(2n+1)\left[2\binom{p}{2p-2j+1}+\binom{p}{2p-2j}\right]B_{jk}m^k. \end{aligned} \tag{18}$$

It is seen that $S_{2p}(n)$ has the form (14), and by comparison of (18) with (14) formulas (15) and (16) follow. Theorem II, therefore, follows by mathematical induction.

THEOREM II(A): *The coefficients B_{pk} can be expressed in the following determinant form*

$$B_{pk} = \frac{(-1)^{p-k}2^{k-1}D_{pk}}{(2p+1)(2p-1)\cdots(2k+1)}, \tag{19}$$

where

$$D_{pk} = \left| b_{ij}^{pk} \right|, \tag{20}$$

i and j take on the values 1, 2, 3, $\cdots$, $(p-k)$, *and*

$$b_{ij}^{pk} = 2\binom{p-j+1}{2i-2j+3} + \binom{p-j+1}{2i-2j+2}. \tag{21}$$

As in the determinant (9), all of the elements of (20) are zero for $j>i+1$. This is valid for $k<p$. For $k=p$ equation (19) gives the correct result if D_{pk} is arbitrarily set equal to 1.

The proof of this theorem is similar to that of Theorem I(A).

It may be true that the determinants (9) and (20) are always positive, since for all values of p up to and including 5 the coefficients $A_{pp}, A_{p,p-1}, A_{p,p-2}, \cdots$, as well as $B_{pp}, B_{p,p-1}, B_{p,p-2}, \cdots$, alternate in sign as may be seen from Table I. Thus far the writer has not found a proof of this, however.

THEOREM II(B): *The coefficients B_{pk} satisfy the following recursion formula*

$$B_{pk} = -\frac{1}{2k+1}\sum_{i=k+1}^{2k} 2^{k-i}B_{pi}\left[2\binom{i}{2i-2k+1} + \binom{i}{2i-2k}\right]. \tag{22}$$

The proof is similar to that of Theorem I(B).

THEOREM III:

$$S_{2p}(n) = \sum_{k=0}^{p} C_{pk}(n+1/2)^{2k+1}, \tag{23}$$

where C_{pk} are rational numbers independent of n which satisfy the relation

$$C_{pp} = \frac{1}{2p+1}, \tag{24}$$

and the recursion formulas

$$C_{pk} = -\sum_{j=1}^{p-1} \frac{1}{2p+1}\cdot\frac{1}{2^{2p-2j}}\binom{2p+1}{2j}C_{jk}, \qquad 1 \leqq k < p, \tag{25}$$

and

$$C_{p0} = -\sum_{j=1}^{p-1} \frac{1}{2p+1}\cdot\frac{1}{2^{2p-2j}}\binom{2p+1}{2j}C_{j0} - \frac{1}{2p+1}\cdot\frac{1}{2^{2p}}. \tag{26}$$

The case $p=0$ is an exception since we have

$$S_0(n) = (n + 1/2) - 1/2.$$

Proof. We begin with the identity

$$(r + 1/2)^{2p+1} - (r - 1/2)^{2p+1} = \sum_{j=0}^{p} 2\binom{2p+1}{2j} r^{2j}(1/2)^{2p-2j+1}.$$

Proceeding as in Theorems I and II the proof is easily obtained.

THEOREM IV:

$$S_{2p-1}(n) = \sum_{k=0}^{p} D_{pk}(n + 1/2)^{2k}, \tag{27}$$

where D_{pk} are rational numbers independent of n that satisfy

$$D_{pp} = \frac{1}{2p}, \tag{28}$$

the recursion formulas

$$D_{pk} = -\frac{1}{2p}\sum_{j=1}^{p-1} (1/2)^{2p-2j}\binom{2p}{2j-1} D_{jk}, \qquad 1 \leqq k < p, \tag{29}$$

and

$$D_{p0} = -\frac{1}{2p}(1/2)^{2p} - \frac{1}{2p}\sum_{j=1}^{p-1} (1/2)^{2p-2j}\binom{2p}{2j-1} D_{j0}. \tag{30}$$

Proof. Starting with the identity

$$(r + 1/2)^{2p} - (r - 1/2)^{2p} = \sum_{j=1}^{p} 2\binom{2p}{2j-1} r^{2j-1}(1/2)^{2p-2j+1} \tag{31}$$

and proceeding as before, the proof is easily obtained.

THEOREM V:

$$R_{2p-1}(2n - 1) = \sum_{k=1}^{p} E_{pk}n^{2k}, \tag{32}$$

where E_{pk} are rational numbers independent of n that satisfy

$$E_{pp} = \frac{2^{2p-2}}{p}, \tag{33}$$

and the recursion formula

$$E_{pk} = -\frac{1}{2p}\sum_{j=1}^{p-1}\binom{2p}{2j-1} E_{jk}, \qquad k < p. \tag{34}$$

Proof. Starting with the identity

$$\frac{[(2r-1)+1]^{2p}}{2^{2p}} - \frac{[(2r-1)-1]^{2p}}{2^{2p}} = \sum_{j=1}^{p} \frac{1}{2^{2p-1}}\binom{2p}{2j-1}(2r-1)^{2j-1} \tag{35}$$

and proceeding as before, the proof follows by mathematical induction.

THEOREM VI:

$$R_{2p}(2n-1) = \sum_{k=0}^{p} F_{pk} n^{2k+1}, \tag{36}$$

where F_{pk} are rational numbers independent of n that satisfy

$$F_{pp} = \frac{2^{2p}}{2p+1}, \tag{37}$$

and the recursion formula

$$F_{pk} = -\frac{1}{2p+1}\sum_{j=0}^{p-1}\binom{2p+1}{2j} F_{jk}. \tag{38}$$

Proof. The starting point is the identity

$$\frac{[(2r-1)+1]^{2p+1}}{2^{2p+1}} - \frac{[(2r-1)-1]^{2p+1}}{2^{2p+1}} = \sum_{j=0}^{p} \frac{1}{2^{2p}}\binom{2p+1}{2j}(2r-1)^{2j} \tag{39}$$

and the proof is similar to that of the preceding theorems.

In the case of theorems III, IV, V and VI, determinant expressions similar to those obtained in Theorems I(A) and II(A) can be found by the same methods. We will not do this, however, because determinant expressions of that type can easily be obtained from the literature; for each coefficient in these theorems is expressible as the product of an algebraic factor and a Bernoulli number,* and every Bernoulli number can be written as a determinant expression.†

It is to be noted also that theorems analogous to I(B) and II(B) can be established in the case of Theorems III–VI inclusive, by the same methods as were used in proving I(B) and II(B).

N. Nielsen* gives a table of the formulas for $S_p(n)$ in powers of n from $p=1$ to 10 inclusive. We have put $S_p(n)$ into the forms indicated in Theorems I–IV inclusive for $p=0$ to $p=10$ inclusive. These formulas are given in Tables I and II. It will be observed from Table I that when we use the form given in Theorems I and II the coefficients are the ratios of fairly small integers.

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* Cf. reference 2.

† Pascal, *Determinanten,* pp. 136–138.

TABLE I

$$m = n(n+1)/2$$

$$S_1(n) = m$$

$$S_2(n) = (2n+1)\frac{m}{3}$$

$$S_3(n) = m^2$$

$$S_4(n) = (2n+1)\left(\frac{2}{5}m^2 - \frac{1}{15}m\right)$$

$$S_5(n) = \frac{4}{3}m^3 - \frac{1}{3}m^2$$

$$S_6(n) = (2n+1)\left(\frac{4}{7}m^3 - \frac{2}{7}m^2 + \frac{1}{21}m\right)$$

$$S_7(n) = 2m^4 - \frac{4}{3}m^3 + \frac{1}{3}m^2$$

$$S_8(n) = (2n+1)\left(\frac{8}{9}m^4 - \frac{8}{9}m^3 + \frac{2}{5}m^2 - \frac{1}{15}m\right)$$

$$S_9(n) = \frac{16}{5}m^5 - 4m^4 + \frac{12}{5}m^3 - \frac{3}{5}m^2$$

$$S_{10}(n) = (2n+1)\left(\frac{16}{11}m^5 - \frac{80}{33}m^4 + \frac{68}{33}m^3 - \frac{10}{11}m^2 + \frac{5}{33}m\right)$$

TABLE II

$$S_0(n)=(n+1/2)-\frac{1}{2}$$

$$S_1(n)=\frac{1}{2}(n+1/2)^2-\frac{1}{4}$$

$$S_2(n)=\frac{1}{3}(n+1/2)^3-\frac{1}{12}(n+1/2)$$

$$S_3(n)=\frac{1}{4}(n+1/2)^4-\frac{1}{8}(n+1/2)^2+\frac{1}{64}$$

$$S_4(n)=\frac{1}{5}(n+1/2)^5-\frac{1}{6}(n+1/2)^3+\frac{7}{240}(n+1/2)$$

$$S_5(n)=\frac{1}{6}(n+1/2)^6-\frac{5}{24}(n+1/2)^4+\frac{7}{96}(n+1/2)^2-\frac{1}{128}$$

$$S_6(n)=\frac{1}{7}(n+1/2)^7-\frac{1}{4}(n+1/2)^5+\frac{7}{48}(n+1/2)^3-\frac{31}{1344}(n+1/2)$$

$$S_7(n)=\frac{1}{8}(n+1/2)^8-\frac{7}{24}(n+1/2)^6+\frac{49}{192}(n+1/2)^4-\frac{31}{384}(n+1/2)^2+\frac{51}{6144}$$

$$S_8(n)=\frac{1}{9}(n+1/2)^9-\frac{1}{3}\cdot(n+1/2)^7+\frac{49}{120}\cdot(n+1/2)^5-\frac{31}{144}(n+1/2)^3+\frac{127}{3840}(n+1/2)$$

$$S_9(n)=\frac{1}{10}(n+1/2)^{10}-\frac{15}{40}(n+1/2)^8+\frac{49}{80}(n+1/2)^6-\frac{155}{320}(n+1/2)^4+\frac{381}{2560}(n+1/2)^2-\frac{31}{2048}$$

$$S_{10}(n)=\frac{1}{11}(n+1/2)^{11}-\frac{5}{12}(n+1/2)^9+\frac{7}{8}(n+1/2)^7-\frac{31}{32}(n+1/2)^5+\frac{127}{256}(n+1/2)^3-\frac{2555}{33792}(n+1/2)$$