A SIMPLE PROOF THAT π IS IRRATIONAL

IVAN NIVEN (BULL AMS **53** (1947), 509)

Let $\pi = a/b$, the quotient of positive integers. We define the polynomials

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$$

the positive integer being specified later. Since $n!f(x)$ has integral coefficients and terms in x of degree not less than n, $f(x)$ and its derivatives $f^{(j)}(x)$ have integral values for $x = 0$; also for $x = \pi = a/b$, since $f(x) = f(a/b - x)$. By elementary calculus we have

$$\frac{d}{dx}\{F'(x)\sin x - F(x)\cos x\} = F''(x)\sin x + F(x)\sin x = f(x)\sin x$$

and

$$\int_0^\pi f(x)\sin x dx = [F'(x)\sin x - F(x)\cos x]_0^\pi = F(\pi) + F(0). \quad (1)$$

Now $F(\pi) + F(0)$ is an *integer*, since $f^{(j)}(\pi)$ and $f^{(j)}(\pi)$ are integers. But for $0 < x < \pi$,

$$0 < f(x)\sin x < \frac{\pi^n a^n}{n!},$$

so that the integral in (1) is *positive but arbitrarily small* for n sufficiently large. Thus (1) is false, and so is our assumption that π is rational.

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