حل تمرین ریاضی عمومی ۱

$$\underbrace{(z_1\bar{z}_2+\overline{z_1\bar{z}_2})^2}_{\text{عدد حقیقی}} \geq 0 \qquad \underbrace{(z_1\bar{z}_2-\overline{z_1\bar{z}_2})^2}_{\text{عدد موهومی}} \leq 0$$

$$(z_1\bar{z}_2+\overline{z_1\bar{z}_2})^2-(z_1\bar{z}_2-\overline{z_1\bar{z}_2})^2 = 4z_1\bar{z}_1z_2\bar{z}_2 = 4|z_1|^2|z_2|^2 \geq 0$$

$$\Rightarrow (z_1\bar{z}_2+\overline{z_1\bar{z}_2})^2 \leq 4|z_1|^2|z_2|^2 \Rightarrow -2|z_1||z_2| \leq z_1\bar{z}_2+\overline{z_1\bar{z}_2} \leq 2|z_1||z_2| \; ($$

بنابه (۱) و (۲)

$$|z_1-z_2|^2 \leq |z_1|^2+|z_2|^2+2|z_1||z_2| = (|z_1|+|z_2|)^2.$$

$$\Rightarrow |z_1-z_2| \leq |z_1|+|z_2|$$

به طور مشابه می‌توان نشان داد $|z_1+z_2| \leq |z_1|+|z_2|$

۳. $|z_1-z_2|^2 = |z_1|^2-2Re(z_1\bar{z}_2)+|z_2|^2$

مشابه به حالت قبل

$$|z_1-\bar{z}_2|^2 = |z_1|^2-(z_1\bar{z}_2+\overline{z_1\bar{z}_2})+|z_2|^2$$

بنابر تمرین ۱ حالت e ، $2Re(z) = z+\bar{z}$ بنابراین

$$|z_1-z_2|^2 = |z_1|^2-2Re(z_1\bar{z}_2)+|z_2|^2$$

۴. $||z_1|-|z_2|| \leq |z_1-z_2|$

مشابه به حالت ۱،

$$|z_1-z_2|^2 = |z_1|^2-(z_1\bar{z}_2+\overline{z_1\bar{z}_2})+|z_2|^2$$

$$-(z_1\bar{z}_2+\overline{z_1\bar{z}_2}) \geq -2|z_1||z_2|$$

بنابراین

$$|z_1-z_2|^2 \geq |z_1|^2-2|z_1||z_2|+|z_2|^2 = (|z_1|-|z_2|)^2 \Rightarrow |z_1-z_2| \geq ||z_1|-|z_2||$$

۵. $\sqrt{2}|z| \geq |Re(z)|+|Im(z)|$

$$\sqrt{2}|z| \geq |Re(z)|+|Im(z)| \iff \sqrt{2}\sqrt{Re(z)^2+Im(z)^2} \geq |Re(z)|+|Im(z)|$$

$$\iff 2(Re(z)^2+Im(z)^2) \geq (|Re(z)|+|Im(z)|)^2 = Re(z)^2+Im(z)^2+2|Re(z)||Im(z)|$$

$$\Longleftrightarrow \quad Re(z)^2 + Im(z)^2 - 2|Re(z)||Im(z)| \geq 0 \quad \Longleftrightarrow \quad (|Re(z)| + |Im(z)|)^2 \geq 0$$

۷. بردارهای ناصفر z_1 و z_2 موازی‌اند اگر و تنها اگر $Im(z_1\bar{z}_2)=0$

فرض می‌کنیم
$z_1 = x_1 + iy_1 \quad , \quad z_2 = x_2 + iy_2$

$$z_1\bar{z}_2 = (x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)$$

فرض می‌کنیم z_1 و z_2 موازی باشند، در این صورت

$$z_1 \times z_2 = 0 \Rightarrow (x_1\vec{i} + y_1\vec{j}) \times (x_2\vec{i} + y_2\vec{j}) = x_1y_2\vec{k} - y_1x_2\vec{k} = 0$$

بنابراین $x_1y_2 = y_1x_2$ از طرفی

$$Im(z_1\bar{z}_2) = x_2y_1 - x_1y_2$$

در نتیجه $Im(z_1\bar{z}_2)=0$.

برعکس اگر $Im(z_1\bar{z}_2)=0$ آنگاه $x_2y_1 = x_1y_2$ و در نتیجه با $z_1 \times z_2 = 0$، بنابراین

z_1 و z_2 موازی هستند.

۸. $|z^n| = |z|^n$

نتیجه را با استفاده از استقرا ثابت می‌کنیم.

$n=1 \qquad |z| = |z|$

فرض می‌کنیم نتیجه برای n برقرار باشد یعنی
$|z^n| = |z|^n$

نشان می‌دهیم نتیجه برای $n+1$ نیز برقرار است.

$$|z^{n+1}| = |z\,z^n| = |z||z^n| = |z||z|^n = |z|^{n+1}$$

حل تمرین ریاضی عمومی ۱

۵- آرگومان اصلی اعداد زیر را تعیین کنید.

a) $1-i$

$|1-i| = \sqrt{2}$ $\quad \cos\theta = \frac{\sqrt{2}}{2}$ $\quad \sin\theta = \frac{-\sqrt{2}}{2}$

$\arg(z) = \{-\frac{\pi}{4} + 2n\pi, n \in \mathbb{Z}\}$ $\quad \operatorname{Arg}(z) = -\frac{\pi}{4}$

b) $-\sqrt{3}+i$

$|-\sqrt{3}+i| = 2$ $\quad \cos\theta = -\frac{\sqrt{3}}{2}$ $\quad \sin\theta = \frac{1}{2}$

$\arg(-\sqrt{3}+i) = \{\frac{5\pi}{6} + 2n\pi, n \in \mathbb{Z}\}$ $\quad \operatorname{Arg}(-\sqrt{3}+i) = \frac{5\pi}{6}$

c) $(-1-i\sqrt{3})^2$

$|-1-i\sqrt{3}| = 2$ $\quad \cos\theta = \frac{-1}{2}$ $\quad \sin\theta = -\frac{\sqrt{3}}{2}$ $\quad \arg(-1-i\sqrt{3}) = \{-\frac{2\pi}{3} + 2n\pi, n \in \mathbb{Z}\}$

با توجه به اینکه $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$ داریم

$\arg(-1-i\sqrt{3})^2 = \{-\frac{4\pi}{3} + 2n\pi, n \in \mathbb{Z}\}$ $\quad \operatorname{Arg}(-i\sqrt{3}-1)^2 = -\frac{4\pi}{3} + 2\pi = \frac{2\pi}{3}$

d) $(1-i)^3$

$|1-i| = \sqrt{2}$ $\quad \cos\theta = \frac{\sqrt{2}}{2}$ $\quad \sin\theta = \frac{-\sqrt{2}}{2}$ $\quad \arg(1-i) = \{\frac{-\pi}{4} + 2n\pi, n \in \mathbb{Z}\}$

$\arg(1-i)^3 = \{-\frac{3\pi}{4} + 2n\pi, n \in \mathbb{Z}\}$ $\quad \operatorname{Arg}(1-i)^3 = \frac{-3\pi}{4}$

e) $\frac{2}{1+i\sqrt{3}} = \frac{2(1-i\sqrt{3})}{4} = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

$\left|\frac{2}{1+i\sqrt{3}}\right| = \left|\frac{1}{2} - \frac{\sqrt{3}}{2} i\right| = 1$ $\quad \cos\theta = \frac{1}{2}$ $\quad \sin\theta = \frac{-\sqrt{3}}{2}$

$\arg\left(\frac{2}{1+i\sqrt{3}}\right) = \{\frac{-\pi}{3} + 2n\pi, n \in \mathbb{Z}\}$ $\quad \operatorname{Arg}\left(\frac{2}{1+i\sqrt{3}}\right) = \frac{-\pi}{3}$

f) $\frac{2}{i-1} = \frac{-2}{\sqrt{2}}(1+i)$

$\left|\frac{2}{i-1}\right| = 2$ $\quad \cos\theta = -\frac{\sqrt{2}}{2}$ $\quad \sin\theta = -\frac{\sqrt{2}}{2}$ $\quad \arg\left(\frac{2}{i-1}\right) = \{-\frac{3\pi}{4} + 2n\pi, n \in \mathbb{Z}\}$ $\quad \operatorname{Arg}\left(\frac{2}{i-1}\right) = \frac{-3\pi}{4}$

g) $\frac{1+i\sqrt{3}}{(1+i)^2} = \frac{1+i\sqrt{3}}{2i} = \frac{-i}{2}(-\sqrt{3}+i) = \frac{\sqrt{3}}{2} - \frac{i}{2}$

$\left|\frac{\sqrt{3}}{2} - \frac{i}{2}\right| = 1 \qquad \cos\theta = \frac{\sqrt{3}}{2} \qquad \sin\theta = \frac{-1}{2}$

$\arg\left(\frac{1+i\sqrt{3}}{(1+i)^2}\right) = \left\{-\frac{\pi}{6} + 2n\pi,\ n\in\mathbb{Z}\right\} \qquad \mathrm{Arg}\left(\frac{1+i\sqrt{3}}{(1+i)^2}\right) = -\frac{\pi}{6}$

h) $(1+i\sqrt{3})(1+i)$

$|1+i\sqrt{3}| = 2 \qquad \cos\theta = \frac{1}{2} \qquad \sin\theta = \frac{\sqrt{3}}{2} \qquad \arg(1+i\sqrt{3}) = \left\{\frac{\pi}{3} + 2n\pi,\ n\in\mathbb{Z}\right\}$

$|1+i| = \sqrt{2} \qquad \cos\theta = \frac{\sqrt{2}}{2} \qquad \sin\theta = \frac{\sqrt{2}}{2} \qquad \arg(1+i) = \left\{\frac{\pi}{4} + 2n\pi,\ n\in\mathbb{Z}\right\}$

$\arg((1+i\sqrt{3})(1+i)) = \arg(1+i\sqrt{3}) + \arg(1+i) = \left\{\frac{7\pi}{12} + 2n\pi,\ n\in\mathbb{Z}\right\}$

$$\mathrm{Arg}((1+i\sqrt{3})(1+i)) = \frac{7\pi}{12}$$

۶. اعداد مختلط زیر را به شکل نمایی نشان دهید.

a) $(\sqrt{3}-i)(1+i\sqrt{3}) = 2\sqrt{3}+2i$

$|\sqrt{3}-i| = 2 \qquad \mathrm{Arg}(\sqrt{3}-i) = -\frac{\pi}{6} \Rightarrow \sqrt{3}-i = 2e^{-\frac{\pi}{6}i}$

$|1+i\sqrt{3}| = 2 \qquad \mathrm{Arg}(1+i\sqrt{3}) = \frac{\pi}{3} \Rightarrow 1+i\sqrt{3} = 2e^{\frac{\pi}{3}i}$

$|2\sqrt{3}+2i| = 4 \qquad \mathrm{Arg}(2\sqrt{3}+2i) = \frac{\pi}{6} \qquad 2\sqrt{3}+2i = 4e^{\frac{\pi}{6}i}$

$(\sqrt{3}-i)(1+i\sqrt{3}) = 2e^{-\frac{\pi}{6}i} \times 2e^{\frac{\pi}{3}i} = 4e^{\frac{\pi}{6}i} = 2\sqrt{3}+2i$

b) $(1+i)^3 = -2+2i$

$|1+i| = \sqrt{2} \qquad \arg(1+i) = \left\{\frac{\pi}{4} + 2n\pi,\ n\in\mathbb{Z}\right\} \Rightarrow \arg(1+i)^3 = \left\{\frac{3\pi}{4} + 2n\pi,\ n\in\mathbb{Z}\right\}$

$(1+i)^3 = (\sqrt{2})^3 \times e^{\frac{3\pi}{4}i} = 2\sqrt{2}\,e^{\frac{3\pi}{4}i}$

$|-2+2i| = 2\sqrt{2} \qquad \arg(-2+2i) = \left\{\frac{3\pi}{4} + 2n\pi,\ n\in\mathbb{Z}\right\} \qquad -2+2i = 2\sqrt{2}\,e^{\frac{3\pi}{4}i}$

$$(1+i)^3 = -2+2i = 2\sqrt{2}\,e^{\frac{3\pi}{4}i}$$

c) $۲i(\sqrt{۳}+i)(۱+i\sqrt{۳}) = -۸$

$|۲i| = ۲$ $\quad \arg(۲i) = \{\frac{\pi}{۲} + ۲n\pi, n \in Z\}$ $\quad ۲i = ۲e^{\frac{\pi}{۲}i}$

$|\sqrt{۳}+i| = ۲$ $\quad \arg(\sqrt{۳}+i) = \{\frac{\pi}{۶} + ۲n\pi, n \in Z\}$ $\quad \sqrt{۳}+i = ۲e^{\frac{\pi}{۶}i}$

$|۱+i\sqrt{۳}| = ۲$ $\quad \arg(۱+i\sqrt{۳}) = \{\frac{\pi}{۳} + ۲n\pi, n \in Z\}$ $\quad ۱+i\sqrt{۳} = ۲e^{\frac{\pi}{۳}i}$

$|-۸| = ۸$ $\quad \arg(-۸) = \{-\pi + ۲n\pi, n \in Z\}$ $\quad -۸ = ۸e^{-i\pi}$

$$۲i(\sqrt{۳}+i)(۱+i\sqrt{۳}) = ۲e^{\frac{\pi}{۲}i} \times ۲e^{\frac{\pi}{۶}i} \times ۲e^{\frac{\pi}{۳}i} = ۸e^{-\pi i} = -۸$$

d) $\frac{۸}{۱+i} = ۴-۴i$

$|۱+i| = \sqrt{۲}$ $\quad \arg(۱+i) = \{\frac{\pi}{۴} + ۲n\pi, n \in Z\}$ $\quad ۱+i = \sqrt{۲}e^{\frac{\pi}{۴}i}$

$|۴-۴i| = ۴\sqrt{۲}$ $\quad \arg(۴-۴i) = \{-\frac{\pi}{۴} + ۲n\pi, n \in Z\}$ $\quad ۴-۴i = ۴\sqrt{۲}e^{-\frac{\pi}{۴}i}$

$$\frac{۸}{۱+i} = \frac{۸}{\sqrt{۲}e^{\frac{\pi}{۴}i}} = ۴\sqrt{۲}e^{-\frac{\pi}{۴}i} = ۴-۴i$$

۷. اعداد زیر را به شکل قطبی نمایش دهید

a) $-۴ = ۴(\cos(\pi) + i\sin(\pi))$

b) $۶-۶i$

$|۶-۶i| = ۶\sqrt{۲}$ $\quad \cos\theta = \frac{\sqrt{۲}}{۲}$ $\quad \sin\theta = -\frac{\sqrt{۲}}{۲}$ $\quad \theta = -\frac{\pi}{۴}$

$$۶-۶i = ۶\sqrt{۲}\left(\cos\left(-\frac{\pi}{۴}\right) + i\sin\left(-\frac{\pi}{۴}\right)\right) = ۶\sqrt{۲}\left(\cos\frac{\pi}{۴} - i\sin\frac{\pi}{۴}\right)$$

c) $-۷i$

$|-۷i| = ۷$ $\quad \cos\theta = ۰$ $\quad \sin\theta = -۱$ $\quad \theta = -\frac{\pi}{۲}$

$$-۷i = ۷\left(\cos\frac{\pi}{۲} - i\sin\frac{\pi}{۲}\right)$$

d) $-2\sqrt{3}-2i$

$|-2\sqrt{3}-2i|=4$ $\quad \cos\theta=-\frac{\sqrt{3}}{2}$ $\quad \sin\theta=-\frac{1}{2}$ $\quad \theta=\frac{\pi}{6}-\pi=-\frac{5\pi}{6}$

$-2\sqrt{3}-2i=4\left(\cos(\frac{5\pi}{6}),-i\sin\frac{5\pi}{6}\right)$

f) $\frac{6}{i+\sqrt{3}}=\frac{3}{2}(\sqrt{3}-i)$

$|\frac{3}{2}(\sqrt{3}-i)|=3$ $\quad \cos\theta=\frac{\sqrt{3}}{2}$ $\quad \sin\theta=-\frac{1}{2}$ $\quad \theta=-\frac{\pi}{6}$

$\frac{6}{i+\sqrt{3}}=3(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6})$

g) $(5+5i)^3$

$|5+5i|=5\sqrt{2}$ $\quad \sin\theta=\frac{\sqrt{2}}{2}$ $\quad \cos\theta=\frac{\sqrt{2}}{2}$ $\quad \theta=\frac{\pi}{4}$

$5+5i=5\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

$(5+5i)^3=(5\sqrt{2})^3(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})$

۸. اعداد زیر را با فرم جبری $a+ib$ نمایش دهید:

a) $e^{i\frac{\pi}{2}}=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i$

b) $4e^{-i\frac{\pi}{2}}=4(\cos(-\frac{\pi}{2})+i\sin(-\frac{\pi}{2}))=-4i$

c) $8e^{i\frac{7\pi}{3}}=8(\cos(\frac{7\pi}{3})+i\sin(\frac{7\pi}{3}))=8(\frac{1}{2}+i\frac{\sqrt{3}}{2})=4(1+\sqrt{3}i)$

d) $-2e^{i\frac{5\pi}{6}}=-2(\cos(\frac{5\pi}{6})+i\sin(\frac{5\pi}{6}))=-2(-\frac{\sqrt{3}}{2}+i\frac{1}{2})=\sqrt{3}-i$

e) $2i\,e^{i\frac{2\pi}{3}}e^{i\pi}=2i(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})(\cos\pi+i\sin\pi)=2i(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)i=1-\sqrt{3}i$

f) $6e^{i\frac{2\pi}{3}}=6(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})=6(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)=3(-1+\sqrt{3}i)$

g) $e^2e^{i\pi}=e^2(\cos\pi+i\sin\pi)=-e^2 i$