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Source: *The American Mathematical Monthly*, Vol. 112, No. 10 (Dec., 2005), pp. 924-925

Published by: Mathematical Association of America

Stable URL: <http://www.jstor.org/stable/30037634>

Accessed: 11/03/2010 09:57

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## On the Moduli of the Zeros of a Polynomial

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Seon-Hong Kim

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A classical result due to Cauchy (see [8, p. 122]) on the distribution of zeros of a polynomial may be stated as follows:

**Theorem 1.** *If  $P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0$  is a polynomial with complex coefficients, then all zeros  $z$  of  $P$  satisfy  $|z| \leq r$ , where  $r$  is the positive solution of the equation*

$$z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \cdots - |a_0| = 0.$$

Díaz-Barrero [4], [5] recently improved this estimate by identifying an annulus containing all the zeros of a polynomial, where the inner and outer radii are expressed in terms of binomial coefficients and Fibonacci numbers. In this note, we use the well-known identity

$$\sum_{k=1}^n C(n, k) = 2^n - 1$$

for the binomial coefficients  $C(n, k) = \binom{n}{k}$  to establish the following enhancement of Cauchy's result:

**Theorem 2.** *Let*

$$P(z) = \sum_{k=0}^n a_k z^k \quad (a_k \neq 0, 1 \leq k \leq n)$$

*be a nonconstant polynomial with complex coefficients. Then all the zeros of  $P(z)$  lie in the annulus*

$$A = \{z: r_1 \leq |z| \leq r_2\}, \tag{1}$$

where

$$r_1 = \min_{1 \leq k \leq n} \left\{ \frac{C(n, k)}{2^n - 1} \left| \frac{a_0}{a_k} \right| \right\}^{1/k}, \quad r_2 = \max_{1 \leq k \leq n} \left\{ \frac{2^n - 1}{C(n, k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}.$$

Theorem 2 appears to be new and improves the estimates in [5], [1], [2], [3], [6], and [7].

**Remark.** For the polynomial  $P(z) = z^3 + 0.1z^2 + 0.3z + 0.7$  (which is used in [5] to establish sharpness of the result there), (1) yields the bounds

$$0.77 \dots \leq |z| \leq 1.19 \dots$$

for any zero  $z$  of  $P$ . These are better than the proposed bounds

$$0.58 \dots \leq |z| \leq 1.23 \dots$$

in [5].

We now prove Theorem 2.

*Proof.* If  $a_0 = 0$ , then  $r_1 = 0$ . If  $a_0 \neq 0$  and  $|z| < r_1$ , we have

$$\begin{aligned} |P(z)| &\geq |a_0| - \sum_{k=1}^n |a_k| |z|^k > |a_0| - \sum_{k=1}^n |a_k| r_1^k = |a_0| \left( 1 - \sum_{k=1}^n \left| \frac{a_k}{a_0} \right| r_1^k \right) \\ &\geq |a_0| \left( 1 - \sum_{k=1}^n \frac{C(n, k)}{2^n - 1} \right) = 0. \end{aligned}$$

Hence  $P(z)$  does not have zeros  $z$  with  $|z| < r_1$ . In view of Theorem 1, it remains to show that  $Q(r_2) \geq 0$ , where

$$Q(z) = |a_n|z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \dots - |a_0|.$$

Now

$$\begin{aligned} Q(r_2) &= |a_n| \left\{ r_2^n - \sum_{k=1}^n \left| \frac{a_{n-k}}{a_n} \right| r_2^{n-k} \right\} \geq |a_n| \left\{ r_2^n - \sum_{k=1}^n \left( \frac{C(n, k)}{2^n - 1} r_2^k \right) r_2^{n-k} \right\} \\ &= |a_n| r_2^n \left( 1 - \sum_{k=1}^n \frac{C(n, k)}{2^n - 1} \right) = 0, \end{aligned}$$

which completes the proof. ■

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