On the Moduli of the Zeros of a Polynomial

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A classical result due to Cauchy (see [**8**, p. 122]) on the distribution of zeros of a polynomial may be stated as follows:

Theorem 1. *If $P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0$ is a polynomial with complex coefficients, then all zeros z of P satisfy $|z| \le r$, where r is the positive solution of the equation*

$$z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \cdots - |a_0| = 0.$$

Díaz-Barrero [**4**], [**5**] recently improved this estimate by identifying an annulus containing all the zeros of a polynomial, where the inner and outer radii are expressed in terms of binomial coefficients and Fibonacci numbers. In this note, we use the well-known identity

$$\sum_{k=1}^{n} C(n,k) = 2^n - 1$$

for the binomial coefficients $C(n,k) = \binom{n}{k}$ to establish the following enhancement of Cauchy's result:

Theorem 2. *Let*

$$P(z) = \sum_{k=0}^{n} a_k z^k \quad (a_k \neq 0, 1 \le k \le n)$$

be a nonconstant polynomial with complex coefficients. Then all the zeros of $P(z)$ lie in the annulus

$$A = \{z : r_1 \le |z| \le r_2\}, \tag{1}$$

where

$$r_1 = \min_{1\le k\le n} \left\{ \frac{C(n,k)}{2^n - 1} \left| \frac{a_0}{a_k} \right| \right\}^{1/k}, \quad r_2 = \max_{1\le k\le n} \left\{ \frac{2^n - 1}{C(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}.$$

Theorem 2 appears to be new and improves the estimates in [**5**], [**1**], [**2**], [**3**], [**6**], and [**7**].

Remark. For the polynomial $P(z) = z^3 + 0.1z^2 + 0.3z + 0.7$ (which is used in **[5]** to establish sharpness of the result there), (1) yields the bounds

$$0.77\cdots \le |z| \le 1.19\cdots$$

for any zero z of P. These are better than the proposed bounds

$$0.58\cdots \le |z| \le 1.23\cdots$$

in **[5]**.

We now prove Theorem 2.

Proof. If $a_0 = 0$, then $r_1 = 0$. If $a_0 \neq 0$ and $|z| < r_1$, we have

$$|P(z)| \ge |a_0| - \sum_{k=1}^{n} |a_k||z|^k > |a_0| - \sum_{k=1}^{n} |a_k| r_1^k = |a_0| \left(1 - \sum_{k=1}^{n} \left|\frac{a_k}{a_0}\right| r_1^k\right)$$
$$\ge |a_0| \left(1 - \sum_{k=1}^{n} \frac{C(n,k)}{2^n - 1}\right) = 0.$$

Hence $P(z)$ does not have zeros z with $|z| < r_1$. In view of Theorem 1, it remains to show that $Q(r_2) \ge 0$, where

$$Q(z) = |a_n|z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \cdots - |a_0|.$$

Now

$$Q(r_2) = |a_n| \left\{ r_2^n - \sum_{k=1}^{n} \left|\frac{a_{n-k}}{a_n}\right| r_2^{n-k} \right\} \ge |a_n| \left\{ r_2^n - \sum_{k=1}^{n} \left(\frac{C(n,k)}{2^n - 1} r_2^k\right) r_2^{n-k} \right\}$$
$$= |a_n| r_2^n \left(1 - \sum_{k=1}^{n} \frac{C(n,k)}{2^n - 1}\right) = 0,$$

which completes the proof. ∎

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