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On the Moduli of the Zeros of a Polynomial

Seon-Hong Kim

A classical result due to Cauchy (see [8, p. 122]) on the distribution of zeros of a polynomial may be stated as follows:

**Theorem 1.** If \( P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0 \) is a polynomial with complex coefficients, then all zeros \( z \) of \( P \) satisfy \( |z| \leq r \), where \( r \) is the positive solution of the equation

\[
|z| - |a_{n-1}| |z|^{n-1} - |a_{n-2}| |z|^{n-2} - \cdots - |a_0| = 0.
\]

Díaz-Barrero [4], [5] recently improved this estimate by identifying an annulus containing all the zeros of a polynomial, where the inner and outer radii are expressed in terms of binomial coefficients and Fibonacci numbers. In this note, we use the well-known identity

\[
\sum_{k=1}^{n} C(n, k) = 2^n - 1
\]

for the binomial coefficients \( C(n, k) = \binom{n}{k} \) to establish the following enhancement of Cauchy's result:

**Theorem 2.** Let

\[
P(z) = \sum_{k=0}^{n} a_k z^k \quad (a_k \neq 0, 1 \leq k \leq n)
\]

be a nonconstant polynomial with complex coefficients. Then all the zeros of \( P(z) \) lie in the annulus

\[
A = \{ z : r_1 \leq |z| \leq r_2 \},
\]

where

\[
r_1 = \min_{1 \leq k \leq n} \left\{ \frac{C(n, k) |a_0|}{2^n - 1} \right\}^{1/k}, \quad r_2 = \max_{1 \leq k \leq n} \left\{ \frac{2^n - 1}{C(n, k)} \right\}^{1/k} \frac{|a_{n-k}|}{|a_n|}.
\]

Theorem 2 appears to be new and improves the estimates in [5], [1], [2], [3], [6], and [7].
Remark. For the polynomial \( P(z) = z^3 + 0.1z^2 + 0.3z + 0.7 \) (which is used in [5] to establish sharpness of the result there), (1) yields the bounds
\[
0.77 \ldots \leq |z| \leq 1.19 \ldots
\]
for any zero \( z \) of \( P \). These are better than the proposed bounds
\[
0.58 \ldots \leq |z| \leq 1.23 \ldots
\]
in [5].

We now prove Theorem 2.

Proof. If \( a_0 = 0 \), then \( r_1 = 0 \). If \( a_0 \neq 0 \) and \( |z| < r_1 \), we have
\[
|P(z)| \geq |a_0| - \sum_{k=1}^{n} |a_k||z|^k > |a_0| - \sum_{k=1}^{n} |a_k|r_1^k = |a_0| \left( 1 - \sum_{k=1}^{n} \frac{|a_k|}{|a_0|} r_1^k \right)
\]
\[
\geq |a_0| \left( 1 - \sum_{k=1}^{n} \frac{C(n, k)}{2^n - 1} \right) = 0.
\]
Hence \( P(z) \) does not have zeros \( z \) with \( |z| < r_1 \). In view of Theorem 1, it remains to show that \( Q(r_2) \geq 0 \), where
\[
Q(z) = |a_n|z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \cdots - |a_0|.
\]
Now
\[
Q(r_2) = |a_n| \left( r_2^n - \sum_{k=1}^{n} \frac{|a_{n-k}|}{|a_n|} r_2^{n-k} \right) \geq |a_n| \left( r_2^n - \sum_{k=1}^{n} \frac{C(n, k)}{2^n - 1} r_2^{n-k} \right)
\]
\[
= |a_n| r_2^n \left( 1 - \sum_{k=1}^{n} \frac{C(n, k)}{2^n - 1} \right) = 0,
\]
which completes the proof.

REFERENCES


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NOTES